

GABOR SYSTEMS AND ALMOST PERIODIC FUNCTIONS

PAOLO BOGGIATTO, CARMEN FERNÁNDEZ AND ANTONIO GALBIS

ABSTRACT. We give a construction of Gabor type frames for suitable separable subspaces of the non-separable Hilbert spaces $AP_2(\mathbb{R})$ of almost periodic functions of one variable. Furthermore we determine a non-countable generalized frame for the whole space $AP_2(\mathbb{R})$. We show furthermore that Bessel-type estimates hold for the AP norm with respect to a countable Gabor system using suitable almost periodic norms of sequences.

1. INTRODUCTION

Almost periodic functions on \mathbb{R} are a natural generalization of usual periodic functions. Their first definition originates with the works of H. Bohr in 1924-1926. Since then a rich theory has been developed by a number of authors, particularly significant among others were the contributions by H. Weyl, N. Wiener, A. S. Besicovich, S. Bochner, V. V. Stepanov, C. de la Vallée Poussin and on LCA groups by J. Von Neumann.

Almost periodic functions found applications in various areas of harmonic analysis, number theory, group theory. In connection with dynamical systems and differential equations they were studied by B. M. Levitan and V. V. Zhikov [15] starting from results of J. Favard. In the more general context of pseudo-differential calculus M. A. Shubin [21] introduced scales of Sobolev spaces of almost periodic functions and recently A. Oliaro, L. Rodino and P. Wahlberg [16] extended some of these results to almost periodic Gevrey classes.

Another recent direction of research is connected with frames theory. As the main spaces of almost periodic functions are non-separable they cannot admit countable frames. The problem then arises in which sense frame-type inequalities are still possible for norm estimations in these spaces. Results in this direction were obtained by F. Galindo [8], and J.R. Partington - B. Unalms [17], [22], using windowed Fourier transform and wavelet transforms, involving therefore Gabor and wavelets

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type frames. Further developments were recently obtained by Y.H. Kim and A. Ron [14], [13], with a fiberization techniques developed by A. Ron and Z. Shen [18], [20] in the context of shift-invariant systems.

This paper has been inspired by the above-mentioned results of Kim-Ron. With different techniques we shall recover and extend some of their results concerning almost periodic norm estimates using Gabor systems. We postpone the description after we have introduced the definitions and properties we need. We refer e.g. to [2], [4], [5] for detailed presentations of the theory of almost periodic functions, and e.g. to [3], [9], [10] for frames theory.

The first type of functions we consider is, according to Bohr, the following.

Definition 1. The space $AP(\mathbb{R})$ of *almost periodic functions* is the set of continuous functions $f : \mathbb{R} \rightarrow \mathbb{C}$ with the property that for every $\epsilon > 0$, there exists $L > 0$, such that every interval of the real line of length greater than L contains a value τ satisfying

$$\sup_t |f(t + \tau) - f(t)| \leq \epsilon.$$

We recall that $AP(\mathbb{R})$ coincides with the uniform norm closure of the space of trigonometric polynomials

$$\sum_{j=1}^N c_j e_{\lambda_j}$$

with $e_{\lambda_j} = e^{i\lambda_j t}$ and $\lambda_j \in \mathbb{R}$, $c_j \in \mathbb{C}$.

The norm $\|\cdot\|_{AP}$ associated with the inner product

$$(f, g)_{AP} = \lim_{T \rightarrow +\infty} (2T)^{-1} \int_{-T}^T f(t) \overline{g(t)} dt$$

makes $AP(\mathbb{R})$ a non-complete, non-separable space.

Definition 2. The completion of $AP(\mathbb{R})$ with respect to this norm is the Hilbert space $AP_2(\mathbb{R})$ (sometimes also indicated as B^2) of *Besicovich almost periodic functions*.

$AP_2(\mathbb{R})$ also coincides with the completion of the trigonometric polynomials with respect to the norm $\|\cdot\|_{AP}$. The set $\{e_\lambda : \lambda \in \mathbb{R}\}$ is a non-countable orthonormal basis of $AP_2(\mathbb{R})$, consequently for every $f \in AP_2(\mathbb{R})$ the expansion $f = \sum_{\lambda} (f, e_\lambda)_{AP} e_\lambda$ holds, where at most countably many terms are different from zero and the norm is given by Parseval's equality $\|f\|_{AP}^2 = \sum_{\lambda \in \mathbb{R}} |(f, e_\lambda)_{AP}|^2$.

We will refer to $(f, e_\lambda)_{AP}$ as the λ -th Fourier coefficient of $f \in AP(\mathbb{R})$ and from now on we will write $\widehat{f}(\lambda) = (f, e_\lambda)_{AP}$. For a given function

$f \in L^1(\mathbb{R})$ we also put

$$\widehat{f}(\lambda) := \int_{\mathbb{R}} f(t) e^{-i\lambda t} dt.$$

Since $AP(\mathbb{R}) \cap L^1(\mathbb{R}) = \{0\}$ there is no possible confusion between the Fourier coefficients of an almost periodic function and the Fourier transform of an integrable function, although we use the same notation for both objects.

As usual the operators of translation and modulation on $L^1(\mathbb{R})$ are defined as

$$T_x \psi(t) = \psi(t - x), \quad M_\omega \psi(t) = e^{i\omega t} \psi(t).$$

Another important concept is the one of almost periodic functions in the sense of Stepanov defined as follows.

Definition 3. The *Stepanov* class S^2 is the closure of the trigonometric polynomials in the amalgam space $W(L^2, L^\infty)$ of those measurable functions f such that

$$\|f\|_{W(L^2, L^\infty)} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} |f(s)|^2 ds \right)^{\frac{1}{2}} < \infty.$$

See e.g. [9], [10] for a presentation of Wiener amalgam spaces. Between the above-mentioned spaces of almost periodic functions the following continuous inclusions hold: $(AP(\mathbb{R}), \|\cdot\|_\infty) \subset (S^2, \|\cdot\|_{W(L^2, L^\infty)}) \subset (AP_2(\mathbb{R}), \|\cdot\|_{AP})$.

Finally an important role in our paper will be played by almost periodic sequences according to the following definition.

Definition 4. A real or complex sequence $(a_j)_{j \in \mathbb{N}}$ is said *almost periodic* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that any set of N consecutive integers contains an integer p satisfying

$$\sup_{n \in \mathbb{N}} |a_{n+p} - a_n| \leq \epsilon.$$

The definition is a discrete version of that of $AP(\mathbb{R})$ and actually almost periodic sequences can be characterized as restrictions to the integers of functions in $AP(\mathbb{R})$. The space of almost periodic sequences, which is denoted by $AP(\mathbb{Z})$, will be endowed with the inner product

$$(a, b)_{AP(\mathbb{Z})} = \lim_{p \rightarrow +\infty} (2p)^{-1} \sum_{n=-p}^p a_n \overline{b_n}$$

and the corresponding norm. For $\lambda \in [0, 2\pi)$, the sequences $\tilde{e}_\lambda = (e^{i\lambda n})_{n \in \mathbb{Z}}$ are a non-countable orthonormal system in $AP(\mathbb{Z})$.

The paper is organized in the following way. In section 2 we show how to construct an almost periodic function from an almost periodic sequence in terms of translates of a fixed continuous function in the

Wiener class. This can be viewed as an extension of the classical periodization lemma. Then, we characterize, for a given $\psi \in L^1(\mathbb{R})$, the continuity of the map

$$S : (AP(\mathbb{R}), \|\cdot\|_{AP}) \rightarrow (AP(\mathbb{Z}), \|\cdot\|_{AP(\mathbb{Z})}), f \mapsto (\langle f, T_{k\alpha}\psi \rangle)_k.$$

The map S is a kind of *analysis map*. Sufficiency is obtained by looking at the continuity of the formal adjoint of S . When ψ is a continuous function in the Wiener amalgam space, the restriction to $AP(\mathbb{Z})$ of this adjoint is

$$a = (a_k)_k \mapsto \sum_{k \in \mathbb{Z}} a_k T_{\alpha k} \psi,$$

which takes values in $AP(\mathbb{R})$ (Theorem 5).

In section 3 we consider Gabor systems $(T_{\alpha k} M_{\beta \ell})_{k, \ell}$ with $\psi \in W_0 \cup (\widehat{W}_0 \cap L^1(\mathbb{R}))$ and prove the estimate

$$\sum_{\ell \in \mathbb{Z}} \|(\langle f, T_{\alpha k} M_{\beta \ell} \psi \rangle)_{k \in \mathbb{Z}}\|_{AP}^2 \leq C \|f\|_{AP}^2$$

for every continuous and almost periodic function f . This is to say that the *analysis* map

$$(AP(\mathbb{R}), \|\cdot\|_{AP}) \rightarrow \ell^2(AP(\mathbb{Z})), f \mapsto (\langle f, T_{\alpha k} M_{\beta \ell} \psi \rangle)_{k, \ell}$$

is continuous. To this end, we first prove the continuity of the formal adjoint map, thus giving a description of what can be considered as the *synthesis* map in this context. This is the content of Theorems 14 and 15.

In section 4 we obtain generalized frames for the non separable space $AP_2(\mathbb{R})$. In particular, we prove that $AP_2(\mathbb{R})$ can be viewed as a direct orthogonal (continuous) sum of separable Hilbert spaces for which we define frames in the usual sense. The frames are described in terms of a given Gabor frame $(T_{\alpha k} M_{\beta \ell})_{k, \ell}$ in $L^2(\mathbb{R})$. Due to the non separability of $AP_2(\mathbb{R})$ the range of the corresponding analysis operator is not longer $\ell^2(\mathbb{Z})$ but the (completion of) the space of square summable $AP(\mathbb{Z})$ -valued sequences, or, alternatively, $L^2(E)$ where $E = [0, \frac{2\pi}{\alpha}) \times \mathbb{Z}$ is endowed with the counting measure. This corresponds to the concept of *generalized frame* in the sense of G. Kaiser [12], S.T Ali - J.P. Antoine - J.P. Gazeau [1], D. Han - J.P. Gabardo [7]. This will permit to connect to Kim-Ron results about almost periodic Gabor frames extending them to the whole space $AP_2(\mathbb{R})$.

Through the paper, α and β are fixed positive constant.

2. ALMOST PERIODICITY AND TRANSLATES

Let us recall that the Wiener space W is the space of functions $\psi \in L^\infty(\mathbb{R})$ such that

$$\|\psi\|_W := \sum_{k \in \mathbb{Z}} \text{ess sup}_{x \in [0,1]} |\psi(x - k)| < \infty.$$

Its closed subspace formed by continuous functions is denoted by W_0 .

The classical periodization Lemma asserts, for fixed $\alpha > 0$, the boundedness of the map $\psi \in W \longrightarrow g = \sum_{k \in \mathbb{Z}} T_{k\alpha} \psi \in L^\infty(\mathbb{R})$ (see e.g. [9], Lemma 6.1.2). We start by a generalization of this Lemma to the almost periodic case, when $\psi \in W_0$.

Theorem 5.

(a) For every $a = (a_k)_{k \in \mathbb{Z}} \in AP(\mathbb{Z})$ and $\psi \in W_0$ we have that

$$g := \sum_{k \in \mathbb{Z}} a_k T_{k\alpha} \psi \in AP(\mathbb{R}).$$

(b) The bilinear map

$$B : AP(\mathbb{Z}) \times W_0 \rightarrow (AP(\mathbb{R}), \|\cdot\|_{AP}), \quad (a, \psi) \mapsto \sum_{k \in \mathbb{Z}} a_k T_{k\alpha} \psi,$$

is continuous.

Proof. (a) We first consider the case $\alpha = 1$. We fix $\epsilon > 0$ and find $L \in \mathbb{N}$ such that

$$\sum_{|k| > L} \sup_{x \in [0,1]} |\psi(x - k)| < \frac{\epsilon}{\|a\|_\infty}.$$

Now choose $N \in \mathbb{N}$ such that among any N consecutive integers there exists an integer p with the property

$$|a_{k+p} - a_k| < \frac{\epsilon}{(2L+1)\|\psi\|_\infty} \quad \forall k \in \mathbb{Z}.$$

Then, for every $t \in [\ell, \ell+1]$,

$$\begin{aligned} \left| g(t) - \sum_{|k| \leq L} a_{k+\ell} T_{k+\ell} \psi(t) \right| &= \left| \sum_{|k| > L} a_{k+\ell} T_{k+\ell} \psi(t) \right| \\ &\leq \|a\|_\infty \sum_{|k| > L} \sup_{x \in [0,1]} |\psi(x - k)| < \epsilon \end{aligned}$$

and also

$$\left| g(t+p) - \sum_{|k| \leq L} a_{k+\ell+p} T_{k+\ell+p} \psi(t+p) \right| < \epsilon$$

for every $p \in \mathbb{Z}$. Then every interval of length N of the real line contain at least one 3ϵ -period of g since, for $t \in [\ell, \ell + 1]$,

$$|g(t+p) - g(t)| \leq 2\epsilon + \sum_{|k| \leq L} |a_{k+\ell+p} - a_{k+\ell}| \cdot |T_{k+\ell}\psi(t)| < 3\epsilon.$$

In the general case, we consider $\varphi(t) = \psi(t\alpha)$ and observe that

$$T_k\varphi\left(\frac{t}{\alpha}\right) = \psi(t - k\alpha) = T_{k\alpha}\psi(t)$$

and

$$g(t) = \left(\sum_{k \in \mathbb{Z}} a_k T_k \varphi\right)\left(\frac{t}{\alpha}\right)$$

is almost periodic.

(b) According to [11, 5.1.4] it suffices to show that B is separately continuous. To this end we first fix $a \in AP(\mathbb{Z})$. For every compact set $K \subset \mathbb{R}$ there is $C > 0$ such that

$$\sup_{t \in K} \left| \sum_{k \in \mathbb{Z}} a_k T_{k\alpha} \psi(t) \right| \leq C \|\psi\|_{W_0}.$$

That is, the map $B(a, \cdot) : W_0 \rightarrow C(\mathbb{R})$ is continuous. Then, an application of the closed graph theorem gives the continuity of

$$B(a, \cdot) : W_0 \rightarrow (AP(\mathbb{R}), \|\cdot\|_\infty).$$

Since $\|f\|_{AP} \leq \|f\|_\infty$ for every $f \in AP(\mathbb{R})$ we conclude that also $B(a, \cdot) : W_0 \rightarrow (AP(\mathbb{R}), \|\cdot\|_{AP})$ is continuous.

We now fix $\psi \in W_0$ and prove the continuity of

$$B(\cdot, \psi) : AP(\mathbb{Z}) \rightarrow (AP(\mathbb{R}), \|\cdot\|_{AP}).$$

We first assume that ψ vanishes outside $[-\alpha M, \alpha M]$ and apply [9, 6.2.2] to get a positive constant C , independent of ψ , such that

$$\left\| \sum_k b_k T_{\alpha k} \psi \right\|_2^2 \leq C^2 \|\psi\|_W^2 \cdot \sum_k |b_k|^2$$

for every finite sequence $(b_k)_k$. We now consider $a = (a_k)_{k \in \mathbb{Z}} \in AP(\mathbb{Z})$ and estimate

$$\begin{aligned} \left\| \sum_{k=-N}^N a_k T_{\alpha k} \psi \right\|_2 &= \left(\int_{-\alpha(M+N)}^{\alpha(M+N)} \left| \sum_{k=-N}^N a_k T_{\alpha k} \psi(t) \right|^2 dt \right)^{\frac{1}{2}} \\ &\geq \left(\int_{-\alpha(M+N)}^{\alpha(M+N)} |B(a, \psi)(t)|^2 dt \right)^{\frac{1}{2}} - \left(\int_{-\alpha(M+N)}^{\alpha(M+N)} \left| \sum_{N < |k| < N+2M} a_k T_{\alpha k} \psi(t) \right|^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} \left| \sum_{N < |k| < N+2M} a_k T_{\alpha k} \psi(t) \right|^2 dt \leq C^2 \|\psi\|_W^2 \cdot \sum_{N < |k| < N+2M} |a_k|^2$$

and

$$\frac{1}{2M} \sum_{N < |k| < N+2M} |a_k|^2$$

is uniformly bounded, it turns out that

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \int_{-\alpha(M+N)}^{\alpha(M+N)} |B(a, \psi)(t)|^2 dt$$

is less than or equal to

$$C^2 \|\psi\|_W^2 \cdot \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N |a_k|^2 = C^2 \|\psi\|_W^2 \cdot \|a\|_{AP}^2.$$

Finally

$$\|B(a, \psi)\|_{AP}^2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |B(a, \psi)(t)|^2 dt \leq \frac{C^2}{\alpha} \|\psi\|_W^2 \cdot \|a\|_{AP}^2$$

where C is a constant which does not depend on ψ or a . In the general case that ψ is not necessarily compactly supported we consider a sequence $\psi_n \in W_0$ of compactly supported functions converging to ψ in W_0 . Since the map

$$B(a, \cdot) : W_0 \rightarrow (AP(\mathbb{R}), \|\cdot\|_{AP})$$

is continuous we get

$$\|B(a, \psi)\|_{AP}^2 = \lim_{n \rightarrow \infty} \|B(a, \psi_n)\|_{AP}^2 \leq \frac{C^2}{\alpha} \|\psi\|_W^2 \cdot \|a\|_{AP}^2.$$

□

As mentioned in Section 1, we denote by e_λ the AP-function $e_\lambda(t) = e^{i\lambda t}$ and by \tilde{e}_λ the AP-sequence $\tilde{e}_\lambda(k) = e^{i\lambda k}$, $k \in \mathbb{Z}$, and observe that $\tilde{e}_{\lambda+2\pi} = \tilde{e}_\lambda$.

Proposition 6. For every $a = (a_k)_{k \in \mathbb{Z}} \in AP(\mathbb{Z})$ and $\psi \in W_0$, the μ -Fourier coefficient of $\sum_k a_k T_{k\alpha} \psi$ is

$$c_\mu = \alpha^{-1} \hat{\psi}(\mu) (a, \tilde{e}_{\mu\alpha})_{AP(\mathbb{Z})}.$$

Proof. (I) We assume at first that ψ is continuous and supported on $[-q\alpha, q\alpha]$. Then, for every $k \in \mathbb{Z}$,

$$\text{supp } T_{k\alpha} \psi \subset [(-q+k)\alpha, (q+k)\alpha].$$

Consequently, for any $N > q$,

$$\begin{aligned}
\int_{-N\alpha}^{N\alpha} \left(\sum_k a_k T_{k\alpha} \psi \right) (t) e^{-i\mu t} dt &= \int_{-N\alpha}^{N\alpha} \left(\sum_{|k| \leq N-q} a_k T_{k\alpha} \psi \right) (t) e^{-i\mu t} dt \\
&= \sum_{|k| \leq N-q} a_k \int_{-\infty}^{\infty} (T_{k\alpha} \psi) (t) e^{-i\mu t} dt \\
&= \widehat{\psi}(\mu) \sum_{|k| \leq N-q} a_k e^{-i\mu k\alpha}.
\end{aligned}$$

Since

$$c_\mu = \lim_{N \rightarrow \infty} \frac{1}{2N\alpha} \int_{-N\alpha}^{N\alpha} \left(\sum_k a_k T_{k\alpha} \psi \right) (t) e^{-i\mu t} dt,$$

the conclusion follows.

(II) We consider now the general case $\psi \in W_0$. According to Theorem 5 (b), for every $a = (a_k)_{k \in \mathbb{Z}} \in AP(\mathbb{Z})$ and $\mu \in \mathbb{R}$, the map

$$\psi \in W_0 \mapsto \mu\text{-Fourier coefficient of } \sum_k a_k T_{k\alpha} \psi$$

is continuous. Since $W_0 \subset L^1(\mathbb{R})$ with continuous inclusion, also

$$\psi \in W_0 \mapsto \alpha^{-1} \widehat{\psi}(\mu)$$

is continuous. Since the continuous functions with compact support are a dense subspace of W_0 , an application of part (I) gives the conclusion. \square

The following observation will play an important role in what follows.

Lemma 7. Let $\psi \in W \cup \widehat{W}_0$ and $\gamma > 0$ be given. Then

$$\sup_\lambda \sum_k \left| \widehat{\psi}(\lambda + \gamma k) \right|^2 < \infty.$$

Proof. The case $\psi \in \widehat{W}_0$ is well-known and follows from [9, 6.1.2] and the fact that W is a Banach algebra with respect to pointwise multiplication. For $\psi \in W$ it suffices to use that $|\widehat{\psi}|^2 \in W_0$ by [6, 2.8], and to apply again [9, 6.1.2]. \square

We will denote by $\langle \cdot, \cdot \rangle$ the usual duality involving L^p spaces,

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t) \overline{g(t)} dt.$$

Lemma 8.

(a) For every $f \in AP(\mathbb{R})$ and $\psi \in L^1(\mathbb{R})$ the sequence

$$(\langle f, T_{k\alpha}\psi \rangle)_{k \in \mathbb{Z}}$$

is almost periodic and the map

$$S : (AP(\mathbb{R}), \|\cdot\|_\infty) \rightarrow (AP(\mathbb{Z}), \|\cdot\|_\infty), \quad f \mapsto (\langle f, T_{k\alpha}\psi \rangle)_k,$$

is continuous.

(b) For every $\lambda \in \mathbb{R}$,

$$(\langle e_\lambda, T_{k\alpha}\psi \rangle)_k = \widehat{\psi}(\lambda) \cdot \tilde{e}_{\lambda\alpha} \in AP(\mathbb{Z}).$$

(c) Let f be trigonometric polynomial. Then

$$(\langle f, T_{k\alpha}\psi \rangle)_k = \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} \left(\sum_p \widehat{f}(\lambda + \frac{2\pi}{\alpha}p) \cdot \overline{\widehat{\psi}(\lambda + \frac{2\pi}{\alpha}p)} \right) \tilde{e}_{\lambda\alpha}.$$

Proof. (a) $\langle f, T_{k\alpha}\psi \rangle = g(k)$ where g is the continuous and almost periodic function $g(t) = (f * \psi)(t\alpha)$. Since

$$|\langle f, T_{k\alpha}\psi \rangle| \leq \|f\|_\infty \cdot \|\psi\|_1 \quad (1)$$

then S is continuous.

(b) Follows from

$$\langle e_\lambda, T_{k\alpha}\psi \rangle = e^{i\lambda k\alpha} \overline{\widehat{\psi}(\lambda)}$$

while (c) is a direct consequence of (b). As $a_\lambda \neq 0$ only for finitely many λ , the sum in (c) is finite. \square

Actually from (1) and the fact that $\|\cdot\|_{AP(\mathbb{Z})} \leq \|\cdot\|_\infty$, we have continuity of the sesquilinear map

$$S : (AP(\mathbb{R}), \|\cdot\|_\infty) \times L^1(\mathbb{R}) \rightarrow (AP(\mathbb{Z}), \|\cdot\|_{AP(\mathbb{Z})}), \quad f \mapsto (\langle f, T_{k\alpha}\psi \rangle)_k.$$

Next we analyze under which conditions on ψ the previous map S is continuous in the case that $AP(\mathbb{R})$ is endowed with the almost periodic norm $\|\cdot\|_{AP}$.

Proposition 9. Given $\psi \in L^1(\mathbb{R})$, the map

$$S : (AP(\mathbb{R}), \|\cdot\|_{AP}) \rightarrow (AP(\mathbb{Z}), \|\cdot\|_{AP(\mathbb{Z})}), \quad f \mapsto (\langle f, T_{k\alpha}\psi \rangle)_k,$$

is continuous if and only if

$$\sup_\lambda \sum_k |\widehat{\psi}(\lambda + 2\pi \frac{k}{\alpha})|^2 < \infty.$$

In particular, this is the case when $\psi \in W$ or $\psi \in L^1(\mathbb{R}) \cap \widehat{W}$.

Proof. Let us assume that S is continuous. For each λ and each $a \in \ell^2$ with finite support we consider the trigonometric polynomial

$$f(t) = \sum_k a_k e^{i(\lambda + 2\pi \frac{k}{\alpha})t}.$$

Since $S(e^{i\mu t}) = \widehat{\psi}(\mu) \tilde{e}_{\mu\alpha}$ then

$$S(f) = \left(\sum_k a_k \widehat{\psi}(\lambda + 2\pi \frac{k}{\alpha}) \right) \tilde{e}_{\lambda\alpha}$$

and

$$\left| \sum_k a_k \widehat{\psi}(\lambda + 2\pi \frac{k}{\alpha}) \right| \leq \|S(f)\| \leq \|S\| \cdot \|f\|_{AP} = \|S\| \cdot \|a\|_{\ell^2},$$

which implies that

$$\sup_{\lambda} \sum_k |\widehat{\psi}(\lambda + 2\pi \frac{k}{\alpha})|^2 \leq \|S\|^2.$$

To prove the converse, consider the map $T : AP(\mathbb{Z}) \rightarrow AP_2(\mathbb{R})$ defined by assigning the Fourier coefficient

$$(T(a), e_{\lambda})_{AP_2} = \widehat{\psi}(\lambda) (a, \tilde{e}_{\lambda\alpha})_{AP(\mathbb{Z})} \quad (2)$$

of $T(a) \in AP_2(\mathbb{R})$, where $a \in AP(\mathbb{Z})$. There is a countable set $\{\lambda_j\}_j$ contained in $[0, \frac{2\pi}{\alpha})$ such that $(a, \tilde{e}_{\lambda\alpha})_{AP(\mathbb{Z})} \neq 0$ implies $\lambda = \lambda_j + 2\pi \frac{k}{\alpha}$ for some $j \in \mathbb{N}$ and $k \in \mathbb{Z}$. Moreover, in this case,

$$(a, \tilde{e}_{\lambda\alpha})_{AP(\mathbb{Z})} = (a, \tilde{e}_{\lambda_j\alpha})_{AP(\mathbb{Z})}.$$

Consequently

$$\begin{aligned} \|T(a)\|_{AP}^2 &= \sum_{\lambda} |\widehat{\psi}(\lambda) (a, \tilde{e}_{\lambda\alpha})|^2 \\ &= \sum_j |(a, \tilde{e}_{\lambda_j\alpha})|^2 \left(\sum_k |\widehat{\psi}(\lambda_j + 2\pi \frac{k}{\alpha})|^2 \right) \\ &\leq C \sum_j |(a, \tilde{e}_{\lambda_j\alpha})|^2 \leq C \|a\|_{AP(\mathbb{Z})}^2, \end{aligned}$$

which shows that T is well defined and continuous. To finish we only need to check that S is the restriction of T^* to $AP(\mathbb{R})$. To this end we first observe that

$$(a, T^*(e_{\lambda}))_{AP_2} = (T(a), e_{\lambda})_{AP_2} = \widehat{\psi}(\lambda) (a, \tilde{e}_{\lambda\alpha})_{AP(\mathbb{Z})},$$

from where it follows

$$T^*(e_{\lambda}) = S(e_{\lambda}).$$

In particular, S and T^* coincide on the trigonometric polynomials, which are a dense subspace of $(AP(\mathbb{R}), \|\cdot\|_\infty)$. Since the map

$$S : (AP(\mathbb{R}), \|\cdot\|_\infty) \rightarrow AP(\mathbb{Z})$$

is continuous, it easily follows that $T^*(f) = S(f)$ for every continuous and almost periodic function $f \in AP(\mathbb{R})$. \square

Remark 10. For every $\psi \in W_0$ the maps

$$T_\psi : AP(\mathbb{Z}) \rightarrow AP(\mathbb{R}), \quad a \mapsto \sum_{k \in \mathbb{Z}} a_k T_{\alpha k} \psi,$$

and

$$S_\psi : AP(\mathbb{R}) \rightarrow AP(\mathbb{Z}), \quad f \mapsto (\langle f, T_{\alpha k} \psi \rangle)_{k \in \mathbb{Z}},$$

are, up to a constant, adjoint to each other.

In fact, for $f = e_\lambda \in AP(\mathbb{R})$ and $b = \tilde{e}_{\mu\alpha}$ we have, by Lemma 8,

$$(S_\psi f, b)_{AP(\mathbb{Z})} = \widehat{\psi}(\lambda) (\tilde{e}_{\lambda\alpha}, \tilde{e}_{\mu\alpha})_{AP(\mathbb{Z})}.$$

Also, Proposition 6 gives

$$(f, T_\psi b)_{AP} = \alpha^{-1} \overline{\widehat{\psi}(\lambda)} (\tilde{e}_{\lambda\alpha}, \tilde{e}_{\mu\alpha})_{AP(\mathbb{Z})}.$$

Consequently

$$S_\psi f = \alpha T_\psi^* f$$

for every trigonometric polynomial f , from where the conclusion follows.

Theorem 11. Let $\psi \in L^1(\mathbb{R})$ satisfies

$$C = \sup_{\lambda} \sum_p \left| \widehat{\psi}(\lambda + \frac{2\pi}{\alpha} p) \right|^2 < \infty. \quad (3)$$

Then

(a) There is an orthogonal system $(h_\lambda)_{\lambda \in [0, \frac{2\pi}{\alpha})}$ in $AP_2(\mathbb{R})$ such that

$$\|(\langle f, T_{k\alpha} \psi \rangle)_{k \in \mathbb{Z}}\|_{AP}^2 = \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} |(f, h_\lambda)_{AP_2}|^2$$

for every $f \in AP(\mathbb{R})$.

(b) If moreover $\psi \in W_0 \cup \widehat{W}_0$ then each $h_\lambda \in AP(\mathbb{R})$.

Proof. (a) Let f be a trigonometric polynomial. Then, from Lemma 8,

$$(\langle f, T_{k\alpha} \psi \rangle)_k = \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} \left(\sum_p \widehat{f}(\lambda + \frac{2\pi}{\alpha} p) \cdot \overline{\widehat{\psi}(\lambda + \frac{2\pi}{\alpha} p)} \right) \tilde{e}_{\lambda\alpha}$$

and

$$\|(\langle f, T_{k\alpha} \psi \rangle)_k\|_{AP}^2 = \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} \left| \sum_p \widehat{f}(\lambda + \frac{2\pi}{\alpha} p) \cdot \overline{\widehat{\psi}(\lambda + \frac{2\pi}{\alpha} p)} \right|^2.$$

Now, fix $\lambda \in [0, \frac{2\pi}{\alpha})$ and consider $c^\lambda = (c_\mu^\lambda)_\mu$ defined as

$$c_\mu^\lambda = \begin{cases} 0 & \text{if } \alpha(\mu - \lambda) \notin 2\pi\mathbb{Z} \\ \widehat{\psi}(\lambda + \frac{2\pi}{\alpha}p) & \text{if } \mu = \lambda + \frac{2\pi}{\alpha}p, p \in \mathbb{Z} \end{cases}$$

By condition (3), there is $h_\lambda \in AP_2(\mathbb{R})$ whose μ -Fourier coefficient is precisely c_μ^λ . Therefore, for each trigonometric polynomial f ,

$$\|(\langle f, T_{k\alpha}\psi \rangle)_{k \in \mathbb{Z}}\|_{AP}^2 = \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} |(f, h_\lambda)_{AP_2}|^2. \quad (4)$$

Moreover, the functions $(h_\lambda)_{\lambda \in [0, \frac{2\pi}{\alpha})}$ are uniformly bounded and form an orthogonal system in $AP_2(\mathbb{R})$ since they have mutually disjoint spectra. By Bessel's inequality, the map

$$f \mapsto \left(\sum_{\lambda \in [0, \frac{2\pi}{\alpha})} |(f, h_\lambda)_{AP_2}|^2 \right)^{\frac{1}{2}}$$

is a continuous seminorm in $AP_2(\mathbb{R})$. Now, Lemma 8 permits to conclude that (4) also holds for every $f \in AP(\mathbb{R})$.

(b) In the case that $\psi \in L^1(\mathbb{R})$ and $\widehat{\psi} \in W_0$ we have that

$$h_\lambda(t) = \sum_p \widehat{\psi}(\lambda + \frac{2\pi}{\alpha}p) e^{i(\lambda + \frac{2\pi}{\alpha}p)t}$$

is a continuous and almost periodic function since

$$\sum_{p \in \mathbb{Z}} \left| \widehat{\psi}(\lambda + \frac{2\pi}{\alpha}p) \right| < \infty.$$

For $\psi \in W_0$ we have

$$h_\lambda(t) = \alpha \sum_{k \in \mathbb{Z}} e^{i\lambda\alpha k} T_{\alpha k} \psi(t),$$

that is,

$$h_\lambda = \alpha \cdot B(\tilde{e}_{\lambda\alpha}, \psi),$$

where B is the bilinear map of Theorem 5. In fact, according to Proposition 6, the Fourier coefficients of $\alpha \cdot B(\tilde{e}_{\lambda\alpha}, \psi)$ and those of h_λ coincide. \square

The identity from Theorem 11 can be extended to functions f in the Stepanov class when $\psi \in W$. We first need an auxiliary result.

Lemma 12. Let $\psi \in W$ and $f \in S^2$ be given. Then

- (a) $f * \psi \in AP(\mathbb{R})$, and
- (b) $\|(\langle f, T_{\alpha k} \psi \rangle)_k\|_{AP(\mathbb{Z})} \leq \|f\|_{W(L^2, L^\infty)} \|\psi\|_W$.

Proof. (a) We put $\psi_k = \psi \chi_{[k, k+1]}$. Then, for every $x \in \mathbb{R}$,

$$\sum_{k \in \mathbb{Z}} \int_{-\infty}^{+\infty} |f(x-t)| \cdot |\psi_k(t)| \, dt$$

is less than or equal to

$$\sum_{k \in \mathbb{Z}} \|f\|_{W(L^2, L^\infty)} \|\psi_k\|_\infty = \|f\|_{W(L^2, L^\infty)} \|\psi\|_W.$$

Consequently, $f * \psi$ is everywhere defined and bounded and

$$\|f * \psi\|_\infty \leq \|f\|_{W(L^2, L^\infty)} \|\psi\|_W.$$

As $P * \psi$ is a trigonometric polynomial for every trigonometric polynomial P , we conclude that $f * \psi$ can be approximated in the sup-norm by a sequence of trigonometric polynomials whenever $f \in S^2$. This gives

(a)

(b) As in Lemma 8 we have that $(\langle f, T_{\alpha k} \psi \rangle)_{k \in \mathbb{Z}} \in AP(\mathbb{Z})$. Moreover

$$\|(\langle f, T_{\alpha k} \psi \rangle)_k\|_{AP(\mathbb{Z})} \leq \|f * \check{\psi}\|_\infty \leq \|f\|_{W(L^2, L^\infty)} \|\psi\|_W.$$

□

The previous result also holds under the less restrictive condition that ψ belongs to the amalgam space $W(L^2, L^1)$.

Corollary 13. Let $\psi \in W$ and $f \in S^2$ be given. Then

$$\|(\langle f, T_{k\alpha} \psi \rangle)_{k \in \mathbb{Z}}\|_{AP}^2 = \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} |(f, h_\lambda)_{AP_2}|^2,$$

where (h_λ) are the functions of Theorem 11.

Proof. It follows from Theorem 11 and Lemma 12. □

3. BESSEL TYPE INEQUALITIES

We recall that $\ell^2(AP(\mathbb{Z}))$ consists of all vector sequences $a = (a^\ell)_{\ell \in \mathbb{Z}}$ where each $a^\ell \in AP(\mathbb{Z})$ and

$$\|a\|^2 := \sum_{\ell \in \mathbb{Z}} \|a^\ell\|_{AP(\mathbb{Z})}^2 < \infty.$$

In this section we will show that, given a Gabor system $(T_{\alpha k} M_{\beta \ell})_{k, \ell}$ with $\psi \in W_0 \cup (\widehat{W_0} \cap L^1(\mathbb{R}))$, the *analysis* map

$$(AP(\mathbb{R}), \| \cdot \|_{AP}) \rightarrow \ell^2(AP(\mathbb{Z})), f \mapsto (\langle f, T_{\alpha k} M_{\beta \ell} \psi \rangle)_{k, \ell},$$

is continuous. This could be obtained as a consequence of general results in [19] with the help of Lemma 17 below. Instead, we consider the natural candidate to be the *synthesis map* and prove its continuity. In our opinion, this gives additional information which can be of

independent interest. Let us recall that given a Bessel Gabor system $(T_{\alpha k} M_{\beta \ell} \psi)_{k, \ell}$ in $L^2(\mathbb{R})$ the synthesis operator is

$$(\alpha_{k, \ell})_{k, \ell} \mapsto \sum_{k, \ell} \alpha_{k, \ell} T_{\alpha k} M_{\beta \ell} \psi$$

for each square summable sequence of scalars $(\alpha_{k, \ell})_{k, \ell}$. Hence, it is reasonable to consider the map

$$a \mapsto \sum_{\ell \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} a_k^\ell T_{\alpha k} M_{\beta \ell} \psi \right)$$

for $a \in \ell^2(AP(\mathbb{Z}))$, having in mind that, for each ℓ , the function

$$\sum_{k \in \mathbb{Z}} a_k^\ell T_{\alpha k} M_{\beta \ell} \psi$$

is almost periodic when $\psi \in W_0$ by Theorem 5.

Theorem 14. Let $\psi \in W_0$ be given. Then the map

$$T : \ell^2(AP(\mathbb{Z})) \rightarrow AP_2(\mathbb{R})$$

defined as

$$T((a^\ell)_{\ell \in \mathbb{Z}}) := \sum_{\ell \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} a_k^\ell T_{\alpha k} M_{\beta \ell} \psi \right),$$

is well defined and continuous. Moreover there is a constant C such that

$$\sum_{\ell \in \mathbb{Z}} \|(\langle f, T_{\alpha k} M_{\beta \ell} \psi \rangle)_{k \in \mathbb{Z}}\|_{AP}^2 \leq C \|f\|_{AP}^2$$

for every $f \in AP(\mathbb{R})$.

Proof. We first assume that ψ vanishes outside $[-\alpha M, \alpha M]$ and apply [9, 6.2.2] to get a positive constant C , independent of ψ , such that

$$\left\| \sum_{\ell, k} b_k^\ell T_{\alpha k} M_{\beta \ell} \psi \right\|_2^2 \leq C^2 \|\psi\|_W^2 \cdot \sum_{\ell, k} |b_k^\ell|^2$$

for every finite sequence $(b_k^\ell)_{k, \ell}$. We now consider $a^\ell = (a_k^\ell)_{k \in \mathbb{Z}} \in AP(\mathbb{Z})$ such that

$$\sum_{\ell \in \mathbb{Z}} \|a^\ell\|_{AP}^2 < \infty$$

and, for every finite subset $J \subset \mathbb{Z}$ we can proceed as in the proof of Theorem 5 (b) to obtain that

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \int_{-\alpha(M+N)}^{\alpha(M+N)} \left| \sum_{\ell \in J} \sum_{k \in \mathbb{Z}} a_k^\ell (T_{\alpha k} M_{\beta \ell} \psi)(t) \right|^2 dt$$

is less than or equal to

$$C^2 \|\psi\|_W^2 \cdot \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{\ell \in J} \sum_{k=-N}^N |a_k^\ell|^2 = C^2 \|\psi\|_W^2 \cdot \sum_{\ell \in J} \|a^\ell\|_{AP}^2.$$

Hence

$$\left\| \sum_{\ell \in J} \sum_{k \in \mathbb{Z}} a_k^\ell T_{\alpha k} M_{\beta \ell} \psi \right\|_{AP}^2 \leq \frac{C^2}{\alpha} \|\psi\|_W^2 \cdot \sum_{\ell \in J} \|a^\ell\|_{AP}^2.$$

As a consequence,

$$\left\{ \sum_{\ell \in J} \left(\sum_{k \in \mathbb{Z}} a_k^\ell T_{\alpha k} M_{\beta \ell} \psi \right) : J \subset \mathbb{Z} \text{ finite} \right\}$$

is a Cauchy net in $AP(\mathbb{R})$ and there exists

$$T((a^\ell)_{\ell \in \mathbb{Z}}) := \sum_{\ell \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} a_k^\ell T_{\alpha k} M_{\beta \ell} \psi \right) \in AP_2(\mathbb{R})$$

with the property that

$$T : \ell^2(AP(\mathbb{Z})) \rightarrow AP_2(\mathbb{R})$$

is a continuous map. Once again, we can proceed as in Theorem 5 to check that the previous statement holds for every $\psi \in W_0$.

To obtain the estimate

$$\sum_{\ell \in \mathbb{Z}} \left\| \left(\langle f, T_{\alpha k} M_{\beta \ell} \psi \rangle \right)_{k \in \mathbb{Z}} \right\|_{AP}^2 \leq C \|f\|_{AP}^2$$

it is enough to show that

$$T^*(f) = \left(\left(\langle f, T_{k\alpha} M_{\ell\beta} \psi \rangle \right)_k \right)_\ell \in \ell^2(AP(\mathbb{Z}))$$

for every $f \in AP(\mathbb{R})$. To this end we denote, for each $\ell' \in \mathbb{Z}$,

$$j_{\ell'} : AP(\mathbb{Z}) \rightarrow \ell^2(AP(\mathbb{Z}))$$

the inclusion $j_{\ell'}(a_k)_k := (a_k^\ell)_{k,\ell}$ where $a_k^\ell = a_k$ for $\ell = \ell'$ and $a_k^\ell = 0$ otherwise, and we define

$$T_{\ell'}(a) := \sum_{k \in \mathbb{Z}} a_k^{\ell'} T_{\alpha k} M_{\beta \ell'} \psi.$$

From Remark 10,

$$T_\ell^*(f) = j_\ell \left(\left(\langle f, T_{k\alpha} M_{\ell\beta} \psi \rangle \right)_k \right).$$

Since

$$T^*(f) = \sum_{\ell \in \mathbb{Z}} T_\ell^*(f),$$

the conclusion follows. \square

Next, we consider the case $\psi \in L^1(\mathbb{R}) \cap \widehat{W}_0$. To deal with this case let us consider the following heuristic argument: first of all every element of $AP_2(\mathbb{R})$ is completely determined by its Fourier coefficients. Given $a \in AP(\mathbb{Z})$ and $\psi \in W_0$ the Fourier coefficients of

$$\sum_{k \in \mathbb{Z}} a_k^\ell T_{\alpha k} M_{\beta \ell} \psi$$

are

$$\widehat{\psi}(\lambda - \ell\beta) \cdot (a^\ell, \tilde{e}_{\lambda\alpha})_{AP(\mathbb{Z})}.$$

Therefore the Fourier coefficients of

$$\sum_{\ell \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} a_k^\ell T_{\alpha k} M_{\beta \ell} \psi \right)$$

should be

$$\sum_{\ell \in \mathbb{Z}} \widehat{\psi}(\lambda - \ell\beta) \cdot (a^\ell, \tilde{e}_{\lambda\alpha})_{AP(\mathbb{Z})}.$$

This argument works when $\psi \in L^1(\mathbb{R}) \cap \widehat{W}_0$.

Theorem 15. Let $\psi \in L^1(\mathbb{R}) \cap \widehat{W}_0$ be given. Then the map

$$T : \ell^2(AP(\mathbb{Z})) \rightarrow AP_2(\mathbb{R}),$$

defined for $a = (a^\ell)_{\ell \in \mathbb{Z}}$ by

$$(Ta, e_\lambda)_{AP_2} = \sum_{\ell \in \mathbb{Z}} \widehat{\psi}(\lambda - \ell\beta) \cdot (a^\ell, \tilde{e}_{\lambda\alpha})_{AP(\mathbb{Z})} \quad (5)$$

is well defined and continuous. Moreover, there is a constant C such that

$$\sum_{\ell \in \mathbb{Z}} \|(\langle f, T_{\alpha k} M_{\beta \ell} \psi \rangle)_{k \in \mathbb{Z}}\|_{AP}^2 \leq C \|f\|_{AP}^2$$

for every $f \in AP(\mathbb{R})$.

Proof. We first observe that

$$\left\{ \lambda \in [0, \frac{2\pi}{\alpha}) : (a^\ell, \tilde{e}_{\lambda\alpha}) \neq 0 \text{ for some } \ell \in \mathbb{Z} \right\}$$

is a countable set, let say $\{\lambda_j\}_{j=1}^\infty$. Moreover

$$\sum_{j=1}^\infty |(a^\ell, \tilde{e}_{\lambda_j\alpha})|^2 = \|a^\ell\|_{AP(\mathbb{Z})}^2.$$

On the other hand, if $(a^\ell, \tilde{e}_{\lambda\alpha}) \neq 0$ for some $\ell \in \mathbb{Z}$ then

$$\lambda = \lambda_j + \frac{2\pi}{\alpha} p$$

for some $j \in \mathbb{N}$ and $p \in \mathbb{Z}$. We put

$$b_{\ell,p}^j = \widehat{\psi}(\lambda_j + \frac{2\pi}{\alpha} p - \ell\beta).$$

According to [9, 6.1.2] we have

$$\sup_{j,\ell} \sum_p |b_{\ell,p}^j| < \infty \quad \text{and} \quad \sup_{j,p} \sum_\ell |b_{\ell,p}^j| < \infty.$$

Schur lemma (see [9, 6.2.1]) implies that, for every j , the matrix $(b_{\ell,p}^j)_{\ell,p}$ defines a bounded operator on $\ell^2(\mathbb{Z})$ and the norm of that operator is bounded by a constant C independent of j . In particular, the sum in (5) is absolutely convergent and

$$\left(\sum_\ell b_{\ell,p}^j (a^\ell, \tilde{e}_{\lambda_j \alpha}) \right)_p \in \ell^2(\mathbb{Z})$$

has norm less than or equal to

$$C \left(\sum_\ell |(a^\ell, \tilde{e}_{\lambda_j \alpha})|^2 \right)^{\frac{1}{2}}.$$

Finally

$$\begin{aligned} \sum_\lambda |(Ta, e_\lambda)_{AP_2}|^2 &= \sum_j \sum_p \left| \sum_\ell b_{\ell,p}^j (a^\ell, \tilde{e}_{\lambda_j \alpha}) \right|^2 \\ &\leq C^2 \sum_j \sum_\ell |(a^\ell, \tilde{e}_{\lambda_j \alpha})|^2 \\ &= C^2 \sum_\ell \|a^\ell\|_{AP(\mathbb{Z})}^2 = C^2 \|a\|^2 < \infty. \end{aligned}$$

This shows that $((Ta, e_\lambda)_{AP_2})$ are the Fourier coefficients of an element $Ta \in AP_2(\mathbb{R})$ and the map T is well-defined and continuous. Consequently

$$T^* : AP_2(\mathbb{R}) \rightarrow \ell^2(B^2(\mathbb{Z}))$$

is a continuous map. Here $B^2(\mathbb{Z})$ denotes the completion of the pre-Hilbert space $AP(\mathbb{Z})$. As in Theorem 14, the proof will be complete if we show that

$$T^*(f) = ((\langle f, T_{k\alpha} M_{\ell\beta} \psi \rangle)_k)_\ell \in \ell^2(AP(\mathbb{Z})) \quad (6)$$

for every $f \in AP(\mathbb{R})$.

First, from Lemma 8 we get

$$(\langle e_\lambda, T_{k\alpha} M_{\ell\beta} \psi \rangle)_k = \widehat{\psi}(\lambda - \ell\beta) \cdot \tilde{e}_{\lambda\alpha} \in AP(\mathbb{Z})$$

and

$$\widehat{\psi}(\lambda - \ell\beta) \cdot (a^\ell, \tilde{e}_{\lambda\alpha})$$

is the inner product in $AP(\mathbb{Z})$ of a^ℓ and $(\langle e_\lambda, T_{k\alpha} M_{\ell\beta} \psi \rangle)_k$. Moreover,

$$\sum_{\ell} \| (\langle e_\lambda, T_{k\alpha} M_{\ell\beta} \psi \rangle)_k \|_{AP(\mathbb{Z})}^2 = \sum_{\ell} \left| \widehat{\psi}(\lambda - \ell\beta) \right|^2 < \infty,$$

which shows that

$$T^*(e_\lambda) = (\langle e_\lambda, T_{k\alpha} M_{\ell\beta} \psi \rangle)_{k,\ell} \in \ell^2(AP(\mathbb{Z})).$$

This proves (6) in the case that f is a trigonometric polynomial. For every $\ell \in \mathbb{Z}$ we denote by π_ℓ the projection onto the ℓ -th coordinate $\pi_\ell : \ell^2(B^2(\mathbb{Z})) \rightarrow B^2(\mathbb{Z})$. Since for every $\ell \in \mathbb{Z}$, both operators

$$(AP(\mathbb{R}), \|\cdot\|_\infty) \rightarrow AP(\mathbb{Z}), f \mapsto (\langle f, T_{k\alpha} M_{\ell\beta} \psi \rangle)_k,$$

and

$$(AP(\mathbb{R}), \|\cdot\|_\infty) \rightarrow B^2(\mathbb{Z}), f \mapsto \pi_\ell(T^*f),$$

are continuous and coincide on the trigonometric polynomials we finally conclude that (6) holds for every $f \in AP(\mathbb{R})$ and the proof is complete. \square

4. GENERALIZED FRAMES FOR ALMOST PERIODIC FUNCTIONS

Although the space $AP_2(\mathbb{R})$ is not separable, in this section we prove that it can be viewed as a direct orthogonal (continuous) sum of separable Hilbert spaces for which we define frames in the usual sense.

The following lemmas will be needed for Theorem 18 which is the main result of this section.

Lemma 16. Let $\psi \in W$ be given, then the map

$$\mathbb{R} \xrightarrow{M} W_0, \quad M(\lambda) = M_{-\lambda}\psi$$

is continuous. If in addition ψ is continuous, the map

$$\mathbb{R} \xrightarrow{T} W_0, \quad T(\lambda) = T_{-\lambda}\psi$$

is also continuous.

Proof. We first prove that M is continuous. In fact, we denote by ψ_k the restriction of ψ to the interval $[k, k+1]$. Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$\sum_{|k| > N} \|\psi_k\|_\infty < \frac{\epsilon}{4}.$$

Now take $\delta > 0$ with the property that $|\lambda - \lambda_0| < \delta$ implies

$$|e^{-i\lambda t} - e^{-i\lambda_0 t}| < \frac{\epsilon}{2\|\psi\|_W}$$

whenever $|t| \leq N+1$. Consequently

$$\|M_{-\lambda}\psi - M_{-\lambda_0}\psi\|_W \leq \frac{\epsilon}{2} + \sum_{|k| \leq N} \text{ess sup}_{t \in [k, k+1]} |e^{-i\lambda t} - e^{-i\lambda_0 t}| |\psi(t)| \leq \epsilon$$

Since W_0 is translation invariant it suffices to check the continuity of T at $\lambda = 0$. Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$\sum_{|k| > N} \sup_{t \in [k, k+1]} |\psi(\lambda + t) - \psi(t)| < \frac{\epsilon}{2}$$

for every λ with $|\lambda| < 1$. Now, since ψ is uniformly continuous, we can take $0 < \delta < 1$ with the property that $|\lambda| < \delta$ implies

$$\sum_{|k| \leq N} \sup_{t \in [k, k+1]} |\psi(\lambda + t) - \psi(t)| < \frac{\epsilon}{2}.$$

Hence $\|T_{-\lambda}\psi - \psi\|_W < \epsilon$ and T is continuous. \square

Lemma 17. Let $\psi \in W \cup \widehat{W}_0$ and $k, p \in \mathbb{Z}$. Then

$$m_{k,p}(\lambda) := \sum_{\ell \in \mathbb{Z}} \widehat{\psi} \left(\lambda + \frac{2\pi}{\alpha} k - \ell\beta \right) \cdot \overline{\widehat{\psi}} \left(\lambda + \frac{2\pi}{\alpha} p - \ell\beta \right)$$

is a continuous function of λ .

Proof. Let X denote one of the Banach spaces W or \widehat{W}_0 , to which ψ belongs. We consider the maps

$$X \xrightarrow{U} \ell^2 \times \ell^2, \quad \ell^2 \times \ell^2 \xrightarrow{V} \ell^1, \quad \mathbb{R} \xrightarrow{M} X$$

defined by

$$U(\varphi) = \left(\left(\widehat{\varphi} \left(\frac{2\pi}{\alpha} k - \ell\beta \right) \right)_{\ell}, \left(\widehat{\varphi} \left(\frac{2\pi}{\alpha} p - \ell\beta \right) \right)_{\ell} \right),$$

and

$$V(a, b) = (a_{\ell} \cdot \overline{b_{\ell}})_{\ell}, \quad M(\lambda) = M_{-\lambda}\psi.$$

Then U and M are continuous by Lemma 7 and Lemma 16, while the continuity of V is obvious. Finally, the map

$$\mathbb{R} \xrightarrow{\Theta} \ell^1, \quad \Theta = V \circ U \circ M$$

is continuous and the conclusion follows. \square

Let $\mathcal{G}(\psi, \alpha, \beta) = (T_{\alpha k} M_{\beta \ell} \psi)_{k, \ell \in \mathbb{Z}}$ be a Gabor system where $\psi \in W \cup (L^1(\mathbb{R}) \cap \widehat{W})$. According to Theorem 11, for every $\ell \in \mathbb{Z}$ there is an orthogonal system $(h_{\lambda, \ell})_{\lambda \in [0, \frac{2\pi}{\alpha})}$ in $AP_2(\mathbb{R})$ such that

$$\|(\langle f, T_{k\alpha} M_{\beta \ell} \psi \rangle)_{k \in \mathbb{Z}}\|_{AP}^2 = \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} |(f, h_{\lambda, \ell})_{AP_2}|^2$$

for every $f \in AP(\mathbb{R})$.

Theorem 18. Let $\psi \in W$ or $\psi \in L^1 \cap \widehat{W}$ and assume that $\mathcal{G}(\psi, \alpha, \beta)$ is a Gabor frame. Then

(a) $\mathcal{G}(\psi, \alpha, \beta)$ is an AP-frame for $AP_2(\mathbb{R})$, i.e. there exist $A, B > 0$ such that

$$A\|f\|_{AP}^2 \leq \sum_{\ell \in \mathbb{Z}} \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} |(f, h_{\lambda, \ell})_{AP_2}|^2 \leq B\|f\|_{AP}^2. \quad (7)$$

for every $f \in AP_2(\mathbb{R})$.

(b) In the case $\psi \in W$ we have

$$A\|f\|_{AP}^2 \leq \sum_{\ell} \|(\langle f, T_{\alpha k} M_{\beta \ell} \psi \rangle)_k\|_{AP}^2 \leq B\|f\|_{AP}^2.$$

for every $f \in S^2$.

Proof. Let us first assume that f is a trigonometric polynomial. According to Lemma 8, for every $\ell \in \mathbb{Z}$,

$$\|(\langle f, T_{k\alpha} M_{\ell\beta} \psi \rangle)_k\|_{AP}^2$$

coincides with

$$\sum_{\lambda \in [0, \frac{2\pi}{\alpha})} \left| \sum_k \widehat{f}(\lambda + \frac{2\pi}{\alpha}k) \cdot \widehat{\psi}(\lambda + \frac{2\pi}{\alpha}k - \ell\beta) \right|^2.$$

All the involved sums are finite, hence

$$\sum_{\ell} \|(\langle f, T_{k\alpha} M_{\ell\beta} \psi \rangle)_k\|_{AP}^2$$

can be written as

$$\sum_{\lambda \in [0, \frac{2\pi}{\alpha})} \sum_{k, p} \widehat{f}(\lambda + \frac{2\pi}{\alpha}k) \overline{\widehat{f}(\lambda + \frac{2\pi}{\alpha}p)} \cdot m_{k, p}(\lambda),$$

where $m_{k, p}(\lambda)$ are as in Lemma 17. On the other hand, as $\mathcal{G}(\psi, \alpha, \beta)$ is a Gabor frame, then

$$\mathcal{G}(\widehat{\psi}, \beta, \alpha)$$

is also a Gabor frame and we can apply [19] (see also [9, 6.3.4]) to conclude that there are positive constants A and B with the property that, for almost every λ , the operator $M(\lambda) : \ell^2 \rightarrow \ell^2$ defined on the finite sequences by

$$\langle M(\lambda)a, b \rangle = \sum_{k, p} m_{k, p}(\lambda) a_p \bar{b}_k$$

is a bounded operator and

$$A\mathbf{I}_{\ell^2} \leq M(\lambda) \leq B\mathbf{I}_{\ell^2}.$$

Let $c = (c_\ell)_{\ell \in \mathbb{Z}}$ be a finite sequence. Then

$$A\|c\|_2^2 \leq \langle M(\lambda)c, c \rangle = \sum_{k, p} m_{k, p}(\lambda) c_p \bar{c}_k \leq B\|c\|_2^2 \quad (8)$$

almost everywhere. Since, from Lemma 17, each $m_{k,p}(\lambda)$ is a continuous function, the inequalities (8) hold for every $\lambda \in \mathbb{R}$. In particular, for every trigonometric polynomial f we obtain

$$\begin{aligned} A \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} \sum_k \left| \widehat{f}\left(\lambda + \frac{2\pi}{\alpha}k\right) \right|^2 &\leq \sum_{\ell} \|(\langle f, T_{k\alpha} M_{\ell\beta} \psi \rangle)_k\|_{AP}^2 \\ &\leq B \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} \sum_k \left| \widehat{f}\left(\lambda + \frac{2\pi}{\alpha}k\right) \right|^2. \end{aligned}$$

That is,

$$A\|f\|_{AP}^2 \leq \sum_{\ell \in \mathbb{Z}} \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} |(f, h_{\lambda, \ell})_{AP_2}|^2 \leq B\|f\|_{AP}^2. \quad (9)$$

Since, for each $\ell \in \mathbb{Z}$, $(h_{\lambda, \ell})_{\lambda}$ is a uniformly bounded orthogonal system and the trigonometric polynomials are dense in $AP_2(\mathbb{R})$ it turns out that each $\varphi_{\ell}(f) := ((f, h_{\lambda, \ell})_{AP_2})_{\lambda}$ defines a continuous map from $AP_2(\mathbb{R})$ into $\ell^2([0, \frac{2\pi}{\alpha}))$ and

$$\sum_{|\ell| \leq N} \|\varphi_{\ell}(f)\|^2 \leq B\|f\|_{AP}^2$$

for every $N \in \mathbb{N}$ and $f \in AP_2(\mathbb{R})$. Consequently,

$$|||f||| := \left(\sum_{\ell} \|\varphi_{\ell}(f)\|^2 \right)^{\frac{1}{2}} = \left(\sum_{\ell \in \mathbb{Z}} \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} |(f, h_{\lambda, \ell})_{AP_2}|^2 \right)^{\frac{1}{2}}$$

defines a continuous norm on $AP_2(\mathbb{R})$. We can conclude that (9) hold for every $f \in AP_2(\mathbb{R})$. This proves (a). In particular,

$$A\|f\|_{AP}^2 \leq \sum_{\ell} \|(\langle f, T_{\alpha k} M_{\beta \ell} \psi \rangle)_k\|_{AP}^2 \leq B\|f\|_{AP}^2. \quad (10)$$

for every continuous almost periodic function f . Moreover, in the case $\psi \in W_0$ we can apply Corollary 13 to conclude that (10) holds for every $f \in S^2$, which proves (b). \square

Remark 19. (a) We observe that

$$\sum_{\ell \in \mathbb{Z}} \sum_{\lambda \in [0, \frac{2\pi}{\alpha})} |(f, h_{\lambda, \ell})_{AP_2}|^2 = \|(f, h_{\lambda, \ell})_{AP}\|_{L^2(E)}^2$$

with $E = [0, \frac{2\pi}{\alpha}) \times \mathbb{Z}$ endowed with discrete (i.e. counting) measure. The statement (a) means that the set of functions $\{h_{\lambda, \ell}\}_{(\lambda, \ell) \in E}$ is a generalized frame in the sense of Kaiser et al. ([1], [7], [12]) as mentioned in the introduction.

(b) If a Gabor frame $\mathcal{G}(\psi, \alpha, \beta)$ defines an AP-frame [14], one can construct for each ℓ the orthogonal system $(h_{\lambda, \ell})_{\lambda \in [0, \frac{2\pi}{\alpha})}$ in $AP_2(\mathbb{R})$ and the estimate (7) holds for each $f \in AP_2(\mathbb{R})$.

(c) Under the hypothesis of Theorem 18, the map

$$AP(\mathbb{R}) \rightarrow \ell^2(AP(\mathbb{Z})), f \mapsto ((\langle f, T_{k\alpha} M_{\ell\beta} \psi \rangle)_k)_\ell$$

can be extended to an isomorphism from $AP_2(\mathbb{R})$ into $\ell^2(AP_2(\mathbb{Z}))$, where $AP_2(\mathbb{Z})$ is the completion of $(AP(\mathbb{Z}), \|\cdot\|_{AP(\mathbb{Z})})$.

Definition 20. For any countable subset $M \subset \mathbb{R}$ we denote by $AP_2(M)$ the closed subspace of $AP_2(\mathbb{R})$ consisting of those $f \in AP_2(\mathbb{R})$ whose spectrum is contained in M .

Then $AP_2(M)$ is a separable Hilbert space and the orthogonal projection onto $AP_2(M)$ of the family of functions $(h_{\lambda, \ell})_{(\lambda, \ell) \in E}$ is a Hilbert frame in $AP_2(M)$. We remark that as particular case of the spaces $AP_2(M)$ we recover the subspace of $AP_2(\mathbb{R})$ of functions with rational almost periodic spectrum. We analyze some special cases.

(a) Assume $\Lambda \subset [0, \frac{2\pi}{\alpha})$ is a countable set and $M = \Lambda + \frac{2\pi}{\alpha}\mathbb{Z}$. Then the orthogonal projection of $(h_{\lambda, \ell})_{(\lambda, \ell) \in E}$ onto $AP_2(M)$ is precisely

$$(h_{\lambda, \ell})_{(\lambda, \ell) \in \Lambda \times \mathbb{Z}}.$$

(b) $M = \{\mu_j : j \in \mathbb{N}\}$ has the property that

$$\mu_i - \mu_j \notin \frac{2\pi}{\alpha}\mathbb{Z}$$

whenever $i \neq j$.

We put

$$\mu_j = \lambda_j + \frac{2\pi}{\alpha}p_j,$$

where $\lambda_j \in [0, \frac{2\pi}{\alpha})$ and $p_j \in \mathbb{Z}$ and denote by $\pi_M : AP_2(\mathbb{R}) \rightarrow AP_2(M)$ the orthogonal projection. Since the spectrum of $h_{\lambda, \ell}$ is contained in $\lambda + \frac{2\pi}{\alpha}\mathbb{Z}$, then $\pi_M(h_{\lambda, \ell}) = 0$ in the case that $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. However, for $\lambda = \lambda_j$ we have

$$\pi_M(h_{\lambda_j, \ell}) = \widehat{\psi}(\mu_j - \ell)e_{\mu_j}.$$

Consequently

$$\left\{ \widehat{\psi}(\mu_j - \ell)e_{\mu_j} : j \in \mathbb{N}, \ell \in \mathbb{Z} \right\}$$

is a Hilbert frame in $AP_2(M)$. We observe that if M is unbounded then no system of the form

$$\left\{ \widehat{\psi}(\mu_j - \ell)e_{\mu_j} : j \in \mathbb{N}, \ell \in F \right\}, \quad F \subset \mathbb{Z} \text{ finite},$$

can be a frame in $AP_2(M)$. In fact,

$$\sum_{\ell \in F} \left| (e_{\mu_j}, \widehat{\psi}(\mu_j - \ell) e_{\mu_j})_{AP} \right|^2$$

is arbitrarily small if $|\mu_j|$ is large enough.

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P. BOGGIATTO: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORINO,
VIA CARLO ALBERTO 10, 10123 TORINO, ITALY
E-mail address: `paolo.boggiatto@unito.it`

C. FERNÁNDEZ: DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD
DE VALENCIA, DOCTOR MOLINER 50, 46100 BURJASOT (VALENCIA), SPAIN
E-mail address: `Carmen.Fdez-Rosell@uv.es`

A. GALBIS: DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE
VALENCIA, DOCTOR MOLINER 50, 46100 BURJASOT (VALENCIA), SPAIN
E-mail address: `Antonio.Galbis@uv.es`